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# HyperDiamond Feynman Checkerboard in 4-dimensional Spacetime

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**Abstract**

A generalized Feynman Checkerboard model is constructed using a 4-dimensional HyperDiamond lattice. The resulting phenomenological model is the $D_4 - D_5 - E_6$ model described in hep-ph/9501252 and quant-ph/9503009.

# Contents

# 1 Introduction.

The 1994 Georgia Tech Ph. D. thesis of Michael Gibbs under David Finkelstein [11] constructed a discrete 4-dimensional spacetime HyperDiamond lattice, here denoted a $4HD$ lattice, in the course of building a physics model.

The 1992 MIT Ph. D. thesis of Hrvoje Hrgovcic under Tommasso Toffoli [12] constructed a discrete 4-dimensional spacetime Minkowski lattice in the course of building a simulation of solutions to the Dirac equation and other equations.
Hrgovcic's lattice is closely related to the $4HD$ lattice.

Both the Gibbs model and the Hrgovcic simulation differ in significant ways from the $D_4 - D_5 - E_6$ model constructed from $3 \times 3$ octonion matrices in hep-ph/9501252 and quant-ph/9503009.

However, both theses have been very helpful with respect to this paper, which generalizes the 2-dimensional Feynman checkerboard [8, 9] to a physically realistic 4-dimensional spacetime.

In this 4-dimensional $4HD$ lattice generalized Feynman HyperDiamond checkerboard, the properties of the particles that move around on the checkerboard are determined by the physically realistic $D_4 - D_5 - E_6$ model constructed from $3 \times 3$ octonion matrices in hep-ph/9501252 and quant-ph/9503009.

The 4-dimensional HyperDiamond $4HD$ checkerboard is related to the 8-dimensional HyperDiamond $8HD = E_8$ in the same way the 4-dimensional physical associative spacetime is related to the 8-dimensional octonionic spacetime in the $D_4 - D_5 - E_6$ model, so the HyperDiamond $4HD$ Feynman checkerboard is fundamentally the discrete version of the $D_4 - D_5 - E_6$ model.

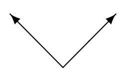
For more about octonions and lattices, see the works of Goeffrey Dixon [3, 4, 5, 6].

## 2 Feynman Checkerboards.

The 2-dimensional Feynman checkerboard [8, 9] is a notably successful and useful representation of the Dirac equation in 2-dimensional spacetime.

To build a Feynman 2-dimensional checkerboard,
start with a 2-dimensional Diamond checkerboard with
two future lightcone links and two past lightcone links at each vertex.

The future lightcone then looks like



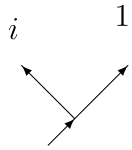
If the 2-dimensional Feynman checkerboard is
coordinatized by the complex plane **C**:
the real axis 1 is identified with the time axis $t$;
the imaginary axis $i$ is identified with the space axis $x$; and
the two future lightcone links are $(1/\sqrt{2})(1+i)$ and $(1/\sqrt{2})(1-i)$.

In cylindrical coordinates $t, r$ with $r^2 = x^2$,
the Euclidian metric is $t^2 + r^2 = t^2 + x^2$ and
the Wick-Rotated Minkowski metric with speed of light $c$ is
$(ct)^2 - r^2 = (ct)^2 - x^2$.

For the future lightcone links to lie on
the 2-dimensional Minkowski lightcone, $c = 1$.

Either link is taken into the other link by complex multiplication by $\pm i$.

Now, consider a path in the Feynman checkerboard.
At a given vertex in the path, denote the future lightcone link in
the same direction as the past path link by 1, and
the future lightcone link in the (only possible) changed direction by $i$.




The Feynman checkerboard rule is that
if the future step at a vertex point of a given path is in a different direction from the immediately preceding step from the past,
then the path at the point of change gets a weight of $-im\epsilon$,
where $m$ is the mass (only massive particles can change directions),and
$\epsilon$ is the length of a path segment.

Here I have used the Gersch [10] convention
of weighting each turn by $-im\epsilon$
rather than the Feynman [8, 9] convention
of weighting by $+im\epsilon$, because Gersch's convention gives a better nonrelativistic limit in the isomorphic 2-dimensional Ising model [10].

HOW SHOULD THIS BE GENERALIZED
TO HIGHER DIMENSIONS?

The 2-dim future light-cone is the 0-sphere $S^{2-2} = S^0 = \{i, 1\}$ ,

with 1 representing a path step to the future in
the same direction as the path step from the past, and

$i$ representing a path step to the future in a
(only 1 in the 2-dimensional Feynman checkerboard lattice)
different direction from the path step from the past.

The 2-dimensional Feynman checkerboard lattice spacetime can be represented by the complex numbers $\mathbf{C}$,
with $1, i$ representing the two future lightcone directions and
$-1, -i$ representing the two past lightcone directions.

Consider a given path in
the Feynman checkerboard lattice 2-dimensional spacetime.

At any given vertex on the path in the lattice 2-dimensional spacetime, the future lightcone direction representing the continuation of the path in the same direction can be represented by 1, and the future lightcone direction representing the (only 1 possible) change of direction can be represented by $i$ since either of the 2 future lightcone directions can be taken into the other by multiplication by $\pm i$, $+$ for a left turn and $-$ for a right turn.

If the path does change direction at the vertex, then the path at the point of change gets a weight of $-im\epsilon$, where $i$ is the complex imaginary, $m$ is the mass (only massive particles can change directions), and $\epsilon$ is the timelike length of a path segment, where the 2-dimensional speed of light is taken to be 1.

Here I have used the Gersch [10] convention of weighting each turn by $-im\epsilon$ rather than the Feynman [8, 9] convention of weighting by $+im\epsilon$, because Gersch's convention gives a better nonrelativistic limit in the isomorphic 2-dimensional Ising model [10].

For a given path, let
$C$ be the total number of direction changes, and
$c$ be the $c$th change of direction, and
$i$ be the complex imaginary representing the $c$th change of direction.

$C$ can be no greater than the timelike checkerboard distance $D$ between the initial and final points.

The total weight for the given path is then

$$\prod_{0 \leq c \leq C} -im\epsilon = (m\epsilon)^C (\prod_{0 \leq c \leq C} -i) = (-im\epsilon)^C \tag{1}$$

The product is a vector in the direction $\pm 1$ or $\pm i$.

Let $N(C)$ be the number of paths with $C$ changes in direction.

The propagator amplitude for the particle to go from the initial vertex to the final vertex is the sum over all paths of the weights, that is the path integral sum over all weighted paths:

$$\sum_{0 \leq C \leq D} N(C)(-im\epsilon)^C \tag{2}$$

The propagator phase is the angle between
the amplitude vector in the complex plane and the complex real axis.

Conventional attempts to generalize the Feynman checkerboard from
2-dimensional spacetime to $k$-dimensional spacetime are based on
the fact that the 2-dimensional future light-cone directions are
the 0-sphere $S^{2-2} = S^0 = \{i, 1\}$.

The $k$-dimensional continuous spacetime lightcone directions are
the $(k-2)$-sphere $S^{k-2}$.

In 4-dimensional continuous spacetime, the lightcone directions are $S^2$.

Instead of looking for a 4-dimensional lattice spacetime, Feynman
and other generalizers went from discrete $S^0$ to continuous $S^2$
for lightcone directions, and then tried to construct a weighting
using changes of directions as rotations in the continuous $S^2$, and
never (as far as I know) got any generalization that worked.

THE $D_4 - D_5 - E_6$ MODEL $4HD$ HYPERDIAMOND GENERALIZATION HAS DISCRETE LIGHTCONE DIRECTIONS.

If the 4-dimensional Feynman checkerboard is coordinatized by the quaternions $\mathbf{Q}$:
the real axis 1 is identified with the time axis $t$;
the imaginary axes $i, j, k$ are identified with the space axes $x, y, z$; and
the four future lightcone links are
$(1/2)(1 + i + j + k)$,
$(1/2)(1 + i - j - k)$,
$(1/2)(1 - i + j - k)$, and
$(1/2)(1 - i - j + k)$.

In cylindrical coordinates $t, r$
with $r^2 = x^2 + y^2 + z^2$,
the Euclidian metric is $t^2 + r^2 = t^2 + x^2 + y^2 + z^2$ and
the Wick-Rotated Minkowski metric with speed of light $c$ is
$(ct)^2 - r^2 = (ct)^2 - x^2 - y^2 - z^2$.

For the future lightcone links to lie on
the 4-dimensional Minkowski lightcone, $c = \sqrt{3}$.

Any future lightcone link is taken into any other future lightcone link by quaternion multiplication by $\pm i$, $\pm j$, or $\pm k$.

For a given vertex on a given path,
continuation in the same direction can be represented by the link 1, and
changing direction can be represented by the
imaginary quaternion $\pm i, \pm j, \pm k$ corresponding to
the link transformation that makes the change of direction.

Therefore, at a vertex where a path changes direction,
a path can be weighted by quaternion imaginaries
just as it is weighted by the complex imaginary in the 2-dimensional case.

If the path does change direction at a vertex, then
the path at the point of change gets a weight of $-im\epsilon$, $-jm\epsilon$, or $-km\epsilon$
where $i, j, k$ is the quaternion imaginary representing the change of direction,

$m$ is the mass (only massive particles can change directions), and
$\sqrt{3}\epsilon$ is the timelike length of a path segment,
where the 4-dimensional speed of light is taken to be $\sqrt{3}$.

For a given path,
let $C$ be the total number of direction changes,
$c$ be the $c$th change of direction, and
$e_c$ be the quaternion imaginary $i, j, k$ representing the $c$th change of direction.

$C$ can be no greater than the timelike checkerboard distance $D$
between the initial and final points.

The total weight for the given path is then

$$\prod_{0 \leq c \leq C} -e_c m\sqrt{3}\epsilon = (m\sqrt{3}\epsilon)^C (\prod_{0 \leq c \leq C} -e_c) \tag{3}$$

Note that since the quaternions are not commutative,
the product must be taken in the correct order.

The product is a vector in the direction $\pm 1$, $\pm i$, $\pm j$, or $\pm k$. and

Let $N(C)$ be the number of paths with $C$ changes in direction.

The propagator amplitude for the particle to go
from the initial vertex to the final vertex is
the sum over all paths of the weights,
that is the path integral sum over all weighted paths:

$$\sum_{0 \leq C \leq D} N(C)(m\sqrt{3}\epsilon)^C (\prod_{0 \leq c \leq C} -e_c) \tag{4}$$

The propagator phase is the angle between
the amplitude vector in quaternionic 4-space and
the quaternionic real axis.

The plane in quaternionic 4-space defined by
the amplitude vector and the quaternionic real axis
can be regarded as the complex plane of the propagator phase.

## 3 HyperDiamond Lattices.

The name "HyperDiamond" was first used by David Finkelstein in our discussions of these structures.

n-dimensional HyperDiamond structures $nHD$ are constructed from $D_n$ lattices.

An n-dimensional HyperDiamond structures $nHD$ is a lattice if and only if $n$ is even.
If $n$ is odd, the $nHD$ structure is only a "packing", not a "lattice", because a nearest neighbor link from an origin vertex to a destination vertex cannot be extended in the same direction to get another nearest neighbor link.

n-dimensional HyperDiamond structures $nHD$ are constructed from $D_n$ lattices.

The lattices of type $D_n$ are n-dimensional checkerboard lattices, that is, the alternate vertices of a $\mathbf{Z}^n$ hypercubic lattice. A general reference on

lattices is Conway and Sloane [1].

For the n-dimensional HyperDiamond lattice construction from $D_n$, Conway and Sloane use an n-dimensional glue vector $[1] = (0.5, ..., 0.5)$ (with n 0.5's).

Consider the 3-dimensional structure $3HD$.

Start with $D_3$, the fcc close packing in 3-space.

Make a second $D_3$ shifted by the glue vector $(0.5, 0.5, 0.5)$.

Then form the union $D_3 \cup ([1] + D_3)$.

That is a 3-dimensional Diamond crystal structure, the familiar 3-dimensional thing for which HyperDiamond lattices are named.

### 3.1 8-dimensional HyperDiamond Lattice.

When you construct an 8-dimensional HyperDiamond $8HD$ lattice, you get $D_8 \cup ([1]+D_8) = E_8$, the fundamental lattice of the octonion structures in the $D_4 - D_5 - E_6$ model described in hep-ph/9501252 [14] and quant-ph/9503009. [15]

The 240 nearest neighbors to the origin in the $E_8$ lattice can be written in 7 different ways using
octonion coordinates with basis

$$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \tag{5}$$

One way is:
16 vertices:

$$\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6, \pm e_7 \tag{6}$$

96 vertices:

$$\begin{array}{c} (\pm 1 \pm e_1 \pm e_2 \pm e_3)/2 \\ (\pm 1 \pm e_2 \pm e_5 \pm e_7)/2 \\ (\pm 1 \pm e_2 \pm e_4 \pm e_6)/2 \\ (\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2 \\ (\pm e_1 \pm e_3 \pm e_4 \pm e_6)/2 \\ (\pm e_1 \pm e_3 \pm e_5 \pm e_7)/2 \end{array} \tag{7}$$

128 vertices:

$$\begin{array}{c} (\pm 1 \pm e_3 \pm e_4 \pm e_7)/2 \\ (\pm 1 \pm e_1 \pm e_5 \pm e_6)/2 \\ (\pm 1 \pm e_3 \pm e_6 \pm e_7)/2 \\ (\pm 1 \pm e_1 \pm e_4 \pm e_7)/2 \\ (\pm e_1 \pm e_2 \pm e_6 \pm e_7)/2 \\ (\pm e_2 \pm e_3 \pm e_4 \pm e_7)/2 \\ (\pm e_1 \pm e_2 \pm e_4 \pm e_5)/2 \\ (\pm e_2 \pm e_3 \pm e_5 \pm e_6)/2 \end{array} \tag{8}$$

That the $E_8$ lattice is, in a sense, fundamentally 4-dimensional can be seen from several points of view:

the $E_8$ lattice nearest neighbor vertices have only 4 non-zero coordinates, like 4-dimensional spacetime with speed of light $c = \sqrt{3}$, rather than 8 non-zero coordinates, like 8-dimensional spacetime with speed of light $c = \sqrt{7}$, so the $E_8$ lattice light-cone structure appears to be 4-dimensional rather than 8-dimensional;

the representation of the $E_8$ lattice by quaternionic icosians, as described by Conway and Sloane [1];

the Golden ratio construction of the $E_8$ lattice from the $D_4$ lattice, which has a 24-cell nearest neighbor polytope (The construction starts with the 24 vertices of a 24-cell, then adds Golden ratio points on each of the 96 edges of the 24-cell, then extends the space to 8 dimensions by considering the algebraicaly independent $\sqrt{5}$ part of the coordinates to be geometrically independent, and finally doubling the resulting 120 vertices in 8-dimensional space (by considering both the $D_4$ lattice and its dual $D_4^*$) to get the 240 vertices of the $E_8$ lattice nearest neighbor polytope (the Witting polytope); and

the fact that the 240-vertex Witting polytope, the $E_8$ lattice nearest neighbor polytope, most naturally lives in 4 complex dimensions, where it is self-dual, rather than in 8 real dimensions.

Some more material on such things can be found at
WWW URL http://www.gatech.edu/tsmith/home.html [13].

In referring to Conway and Sloane [1], bear in mind that they use the convention (usual in working with lattices) that the norm of a lattice distance is the square of the length of the lattice distance.

### 3.2 4-dimensional HyperDiamond Lattice.

The 4-dimensional HyperDiamond lattice $4HD$ is
$4HD = D_4 \cup ([1] + D_4)$.

The 4-dimensional HyperDiamond $4HD = D_4 \cup ([1] + D_4)$ is the $\mathbf{Z}^4$ hypercubic lattice with null edges.

It is the lattice that Michael Gibbs [11] uses in his Ph. D. thesis advised by David Finkelstein.

The 8 nearest neighbors to the origin in the 4-dimensional HyperDiamond $4HD$ lattice can be written in octonion coordinates as:

$$\begin{array}{c} (1+i+j+k)/2 \\ (1+i-j-k)/2 \\ (1-i+j-k)/2 \\ (1-i-j+k)/2 \\ (-1-i+j+k)/2 \\ (-1+i-j+k)/2 \\ (-1+i+j-k)/2 \\ (-1-i-j-k)/2 \end{array} \tag{9}$$

Here is an explicit construction of the 4-dimensional HyperDiamond $4HD$ lattice nearest neighbors to the origin.

START WITH THE 24 VERTICES OF A 24-CELL $D_4$:

$$\begin{array}{cccc}
+1 & +1 & 0 & 0 \\
+1 & 0 & +1 & 0 \\
+1 & 0 & 0 & +1 \\
+1 & -1 & 0 & 0 \\
+1 & 0 & -1 & 0 \\
+1 & 0 & 0 & -1 \\
-1 & +1 & 0 & 0 \\
-1 & 0 & +1 & 0 \\
-1 & 0 & 0 & +1 \\
-1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
0 & +1 & +1 & 0 \\
0 & +1 & 0 & +1 \\
0 & +1 & -1 & 0 \\
0 & +1 & 0 & -1 \\
0 & -1 & +1 & 0 \\
0 & -1 & 0 & +1 \\
0 & -1 & -1 & 0 \\
0 & -1 & 0 & -1 \\
0 & 0 & +1 & +1 \\
0 & 0 & +1 & -1 \\
0 & 0 & -1 & +1 \\
0 & 0 & -1 & -1
\end{array} \tag{10}$$

SHIFT THE LATTICE BY A GLUE VECTOR, BY ADDING

$$0.5 \quad 0.5 \quad 0.5 \quad 0.5 \tag{11}$$

TO GET 24 MORE VERTICES $[1] + D_4$:

$$\begin{array}{cccc}
+1.5 & +1.5 & 0.5 & 0.5 \\
+1.5 & 0.5 & +1.5 & 0.5 \\
+1.5 & 0.5 & 0.5 & +1.5 \\
+1.5 & -0.5 & 0.5 & 0.5 \\
+1.5 & 0.5 & -0.5 & 0.5 \\
+1.5 & 0.5 & 0.5 & -0.5 \\
-0.5 & +1.5 & 0.5 & 0.5 \\
-0.5 & 0.5 & +1.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & +1.5 \\
-0.5 & -0.5 & 0.5 & 0.5 \\
-0.5 & 0.5 & -0.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & +1.5 & +1.5 & 0.5 \\
0.5 & +1.5 & 0.5 & +1.5 \\
0.5 & +1.5 & -0.5 & 0.5 \\
0.5 & +1.5 & 0.5 & -0.5 \\
0.5 & -0.5 & +1.5 & 0.5 \\
0.5 & -0.5 & 0.5 & +1.5 \\
0.5 & -0.5 & -0.5 & 0.5 \\
0.5 & -0.5 & 0.5 & -0.5 \\
0.5 & 0.5 & +1.5 & +1.5 \\
0.5 & 0.5 & +1.5 & -0.5 \\
0.5 & 0.5 & -0.5 & +1.5 \\
0.5 & 0.5 & -0.5 & -0.5
\end{array} \tag{12}$$

FOR THE NEW COMBINED LATTICE $D_4 \cup ([1] + D_4)$, THESE ARE 6 OF THE NEAREST NEIGHBORS TO THE ORIGIN:

$$\begin{array}{cccc} -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \end{array} \tag{13}$$

HERE ARE 2 MORE THAT COME FROM ADDING THE GLUE VECTOR TO LATTICE VECTORS THAT ARE NOT NEAREST NEIGHBORS OF THE ORIGIN:

$$\begin{array}{cccc} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 \end{array} \tag{14}$$

THEY COME, RESPECTIVELY, FROM ADDING THE GLUE VECTOR TO:

THE ORIGIN

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \tag{15}$$

ITSELF;

AND

THE LATTICE POINT

$$-1 \quad -1 \quad -1 \quad -1 \tag{16}$$

WHICH IS SECOND ORDER, FROM

$$\begin{array}{cccc} -1 & -1 & 0 & 0 \\ \multicolumn{4}{l}{plus} \\ 0 & 0 & 0 & 0 \end{array} \tag{17}$$

FROM

$$\begin{array}{cccc} -1 & 0 & -1 & 0 \\ \multicolumn{4}{l}{plus} \\ 0 & -1 & 0 & -1 \end{array} \tag{18}$$

OR FROM

$$\begin{array}{cccc} -1 & 0 & 0 & -1 \\ \multicolumn{4}{l}{plus} \\ 0 & -1 & -1 & 0 \end{array} \tag{19}$$

### 3.3 From 8 to 4 Dimensions.

Dimensional reduction of the $8HD = E_8$ lattice spacetime to 4-dimensional spacetime reduces each of the $D_8$ lattices in the

$$E_8 = 8HD = D_8 \cup ([1] + D_8)$$

lattice to $D_4$ lattices.

Therefore, we should get a 4-dimensional HyperDiamond

$$4HD = D_4 \cup ([1] + D_4)$$

lattice.

To see this, start with $E_8 = 8HD = D_8 \cup ([1] + D_8)$.

We can write:

$$D_8 = \{(D_4, 0, 0, 0, 0)\} \cup \{(0, 0, 0, 0, D_4)\} \cup \{(1, 0, 0, 0, 1, 0, 0, 0) + (D_4, D_4)\} \tag{20}$$

The third term is the diagonal term of
an orthogonal decomposition of $D_8$, and

the first two terms are the orthogonal
associative physical 4-dimensional spacetime
and
coassociative 4-dimensional internal symmetry space
as described in
*Standard Model plus Gravity from*
*Octonion Creators and Annihilators*,
quant-th/9503009 [15].

Now, we see that the orthogonal decomposition of 8-dimensional spacetime into 4-dimensional associative physical spacetime plus 4-dimensional internal symmetry space gives a decomposition of $D_8$ into $D_4 \oplus D_4$.

Since $E_8 = D_8 \cup ([1] + D_8)$, and
since $[1] = (0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)$ can
be decomposed by

$$[1] = (0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0.5, 0.5, 0.5, 0.5) \tag{21}$$

we have

$$\begin{aligned}
E_8 &= D_8 \cup ([1] + D_8) \qquad (22) \\
&= ((D_4, 0, 0, 0, 0) \oplus (0, 0, 0, 0, D_4)) \\
&\quad \cup \\
&\quad (((0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0.5, 0.5, 0.5, 0.5)) \\
&\quad + \\
&\quad ((D_4, 0, 0, 0, 0) \oplus (0, 0, 0, 0, D_4))) \\
&= ((D_4, 0, 0, 0, 0) \cup ((0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0) + (D_4, 0, 0, 0, 0))) \\
&\quad \oplus \\
&\quad ((0, 0, 0, 0, D_4) \cup ((0, 0, 0, 0, 0.5, 0.5, 0.5, 0.5) + (0, 0, 0, 0, D_4)))
\end{aligned}$$

Since $4HD$ is $D_4 \cup ([1] + D_4)$,

$$E_8 = 8HD = 4HD_a \oplus 4HD_{ca} \tag{23}$$

where $4HD_a$ is the 4-dimensional associative physical spacetime and
$4HD_{ca}$ is the 4-dimensional coassociative internal symmetry space.

# 4 Internal Symmetry Space.

$4HD_{ca}$ is the 4-dimensional coassociative Internal Symmetry Space of the 4-dimensional HyperDiamond Feynman checkerboard version of the $D_4 - D_5 - E_6$ model.

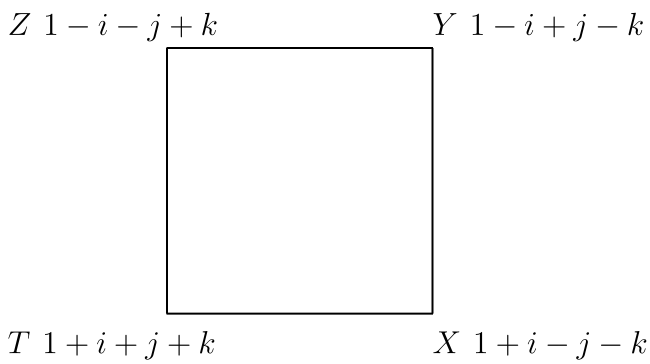
Physically, the $4HD_{ca}$ Internal Space should be thought of as a space "inside" each vertex of the $4HD_a$ Feynman checkerboard spacetime, sort of like a Kaluza-Klein structure.

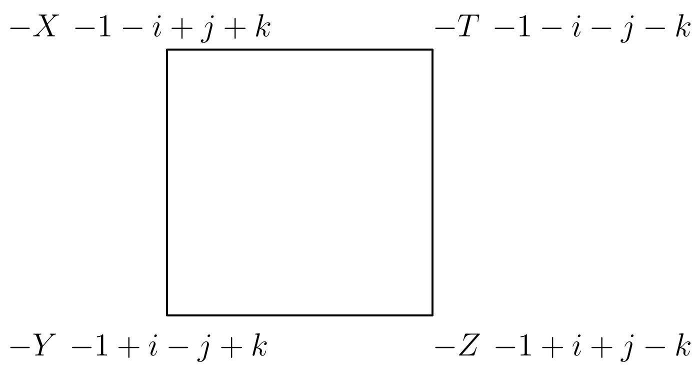
The 4 dimensions of the $4HD_{ca}$ Internal Symmetry Space are:
electric charge;
red color charge;
green color charge; and
blue color charge.

Each vertex of the $4HD_{ca}$ lattice has 8 nearest neighbors, connected by lightcone links. They have the algebraic structure of the 8-element quaternion group $< 2, 2, 2 >$. [2]

Each vertex of the $4HD_{ca}$ lattice has 24 next-to-nearest neighbors, connected by two lightcone links. They have the algebraic structure of the 24-element binary tetrahedral group $< 3, 3, 2 >$ that is associated with the 24-cell and the $D_4$ lattice. [2]

The 1-time and 3-space dimensions of the $4HD_a$ spacetime can be represented by the 4 future lightcone links and the 4 past lightcone links as in the following pair of "Square Diagrams" of the 4 lines connecting the future ends of the 4 future lightcone links and of the 4 lines connecting the past ends of the 4 past lightcone links:



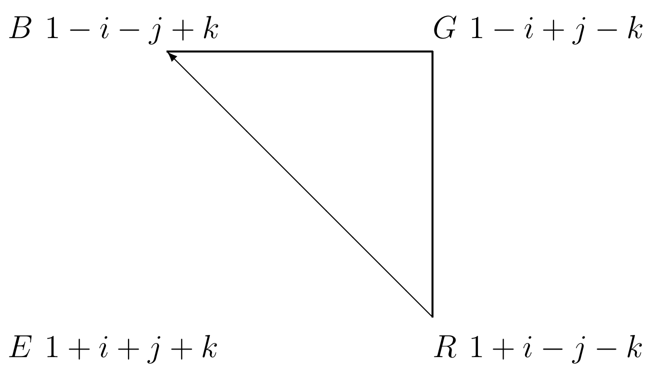



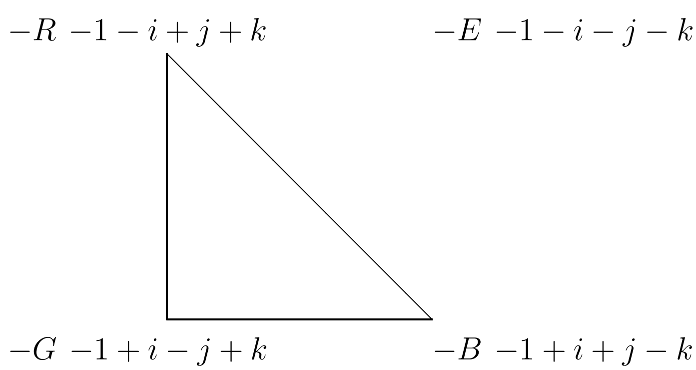

The 8 links $\{\mathbf{T}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, -\mathbf{T}, -\mathbf{X}, -\mathbf{Y}, -\mathbf{Z}\}$ correspond to the 8 root vectors of the $Spin(5)$ de Sitter gravitation gauge group, which has an 8-element Weyl group $S_2^2 \times S_2$.
The symmetry group of the 4 links of the future lightcone is $S_4$, the Weyl group of the 15-dimensional Conformal group $SU(4) = Spin(6)$.
10 of the 15 dimensions make up the de Sitter $Spin(5)$ subgroup, and
the other 5 fix the "symmetry-breaking direction" and scale of the Higgs mechanism (for more on this, see [14] and
WWW URL http://www.gatech.edu/tsmith/cnfGrHg.html).

Similarly, the E-electric and RGB-color dimensions of the $4HD_{ca}$ Internal Symmetry Space can be represented by the 4 future lightcone links and the 4 past lightcone links.

However, in the $4HD_{ca}$ Internal Symmetry Space the E Electric Charge should be treated as independent of the RGB Color Charges.

As a result the following pair of "Square Diagrams" look more like "Triangle plus Point Diagrams".



The 2+6 links $\{\mathbf{E}, -\mathbf{E}; \mathbf{R}, \mathbf{G}, \mathbf{B}, -\mathbf{R}, -\mathbf{G}, -\mathbf{B}\}$ correspond to:

the 2 root vectors of the weak force $SU(2)$, which has a 2-element Weyl group $S_2$; and

the 6 root vectors of the color force $SU(3)$, which has a 6-element Weyl group $S_3$.

# 5 Protons, Pions, and Physical Gravitons.

In his 1994 Georgia Tech Ph. D. thesis under David Finkelstein, *Spacetime as a Quantum Graph*, Michael Gibbs [11] describes some 4-dimensional HyperDiamond lattice structures, that he considers likely candidates to represent physical particles.

The terminology used by Michael Gibbs in his thesis [11] is useful with respect to the model he constructs. Since his model is substantially different from my HyperDiamond Feynman checkerboard in some respects, I use a different terminology here. However, I want to make it clear that I have borrowed these particular structures from his thesis.

Three useful HyperDiamond structures are:

3-link Rotating Propagator, useful for building a proton out of 3 quarks;

2-link Exchange Propagator, useful for building a pion out of a quark and an antiquark; and

4-link Propagator, useful for building a physical spin-2 physical graviton out of $Spin(5)$ Gauge bosons..

In the 2-dimensional Feynman checkerboard, there is only one massive particle, the electron.

What about the $D_4 - D_5 - E_6$ model, or any other model that has different particles with different masses?

In the context of Feynman checkerboards, mass is just the amplitude for a particle to have a change of direction in its path.

More massive particles will change direction more often.

In the $D_4 - D_5 - E_6$ model, the $4HD$ Feynman checkerboard fundamental path segment length $\epsilon$ of any particle the Planck length $L_{PL}$.

However, in the sum over paths for a particle of mass $m$,
it is a useful approximation to consider the path segment length to be the

Compton wavelength $L_m$ of the mass $m$,

$$L_m = h/mc$$

That is because the distances between direction changes in the vast bulk of the paths will be at least $L_m$, and
those distances will be approximately integral multiples of $L_m$,
so that $L_m$ can be used as the effective path segment length.

This is an important approximation because the Planck length $L_{PL}$ is about $10^{-33}$ cm,
while the effective length $L_{100GeV}$ for a particle of mass 100 $GeV$ is about $10^{-16}$ cm.

In this section, the $4HD$ lattice is given quaternionic coordinates.
The orgin 0 designates the beginning of the path.
The 4 future lightcone links from the origin are given the coordinates
$1+i+j+k$, $1+i-j-k$, $1-i+j-k$, $1-i-j+k$

The path of a "Particle" at rest in space, moving 7 steps in time, is denoted by

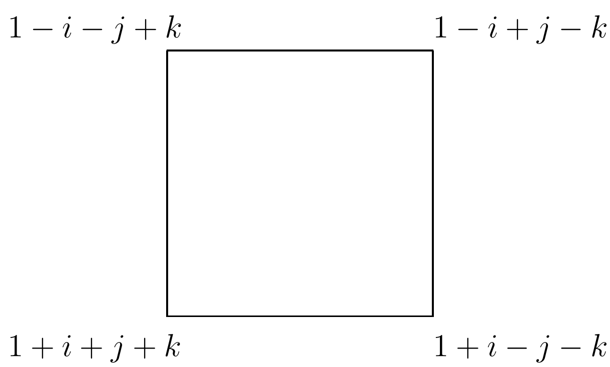
| *Particle* |
|---|
| 0 |
| 1 |
| 2 |
| 3 |
| 4 |
| 5 |
| 6 |
| 7 |

Note that since the $4HD$ speed of light is $\sqrt{3}$, the path length is $7\sqrt{3}$.

The path of a "Particle" moving along a lightcone path in the $1+i+j+k$ direction for 7 steps with no change of direction is

| $Particle$ |
|---|
| $0$ |
| $1+i+j+k$ |
| $2+2i+2j+2k$ |
| $3+3i+3j+3k$ |
| $4+4i+4j+4k$ |
| $5+5i+5j+5k$ |
| $6+6i+6j+6k$ |
| $7+7i+7j+7k$ |

At each step in either path, the future lightcone can be represented by a "Square Diagram" of lines connecting the future ends of the 4 future lightcone links leading from the vertex at which the step begins.




In the following subsections, protons, pions, and physical gravitons will be represented by multiparticle paths.
The multiple particles representing protons, pions, and physical gravitons will be shown on sequences of such Square Diagrams, as well as by a sequence of coordinates.

The coordinate sequences will be given only for a representative sequence of timelike steps, with no space movement, because
the notation for a timelike sequence is clearer and
it is easy to transform a sequence of timelike steps into a sequence of lightcone link steps, as shown above.

Only in the case of gravitons will it be useful to explicitly discuss a path that moves in space as well as time.

## 5.1 3-Quark Protons.

In this $4HD$ Feynman checkerboard version of the $D_4 - D_5 - E_6$ model, protons are made up of 3 valence first generation quarks, (two up quarks and one down quark), with one Red, one Green, and one Blue in color.
The proton bound state of 3 valence quarks has a soliton structure (see WWW URL http://www.gatech.edu/tsmith/SolProton.html [13]).

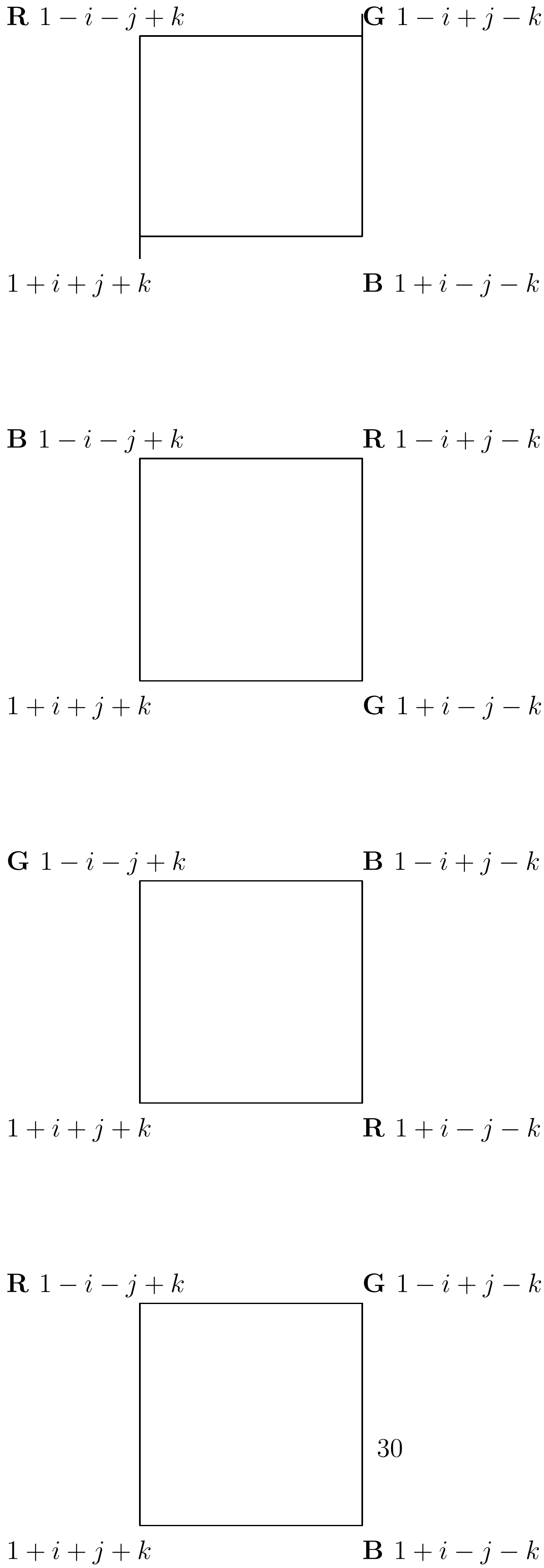
The complicated structure of the sea quarks and binding gluons can be ignored in a $4HD$ Feynman checkerboard approximation that uses the R, G, and B valence quarks and path length $L_{313GeV}$ that is the Compton wavelength of the first generation quark constituent mass.

The $4HD$ structure used to approximate the proton is the 3-link Rotating Propagator of the thesis of Michael Gibbs [11].

Here is a coordinate sequence representation of the approximate $4HD$ Feynman checkerboard path of a proton:

| $R - Quark$ | $G - Quark$ | $B - Quark$ |
|---|---|---|
| $1 + i - j - k$ | $1 - i + j - k$ | $1 - i - j + k$ |
| $2 - i + j - k$ | $2 - i - j + k$ | $2 + i - j - k$ |
| $3 - i - j + k$ | $3 + i - j - k$ | $2 - i + j - k$ |
| $4 + i - j - k$ | $4 - i + j - k$ | $4 - i - j + k$ |
| $5 - i + j - k$ | $5 - i - j + k$ | $5 + i - j - k$ |
| $6 - i - j + k$ | $6 + i - j - k$ | $6 - i + j - k$ |
| $7 + i - j - k$ | $7 - i + j - k$ | $7 - i - j + k$ |

The following page contains a Square Diagram representation of the approximate $4HD$ Feynman checkerboard path of a proton:


**R** $1 - i - j + k$
**G** $1 - i + j - k$
$1 + i + j + k$
**B** $1 + i - j - k$
**B** $1 - i - j + k$
**R** $1 - i + j - k$
$1 + i + j + k$
**G** $1 + i - j - k$
**G** $1 - i - j + k$
**B** $1 - i + j - k$
$1 + i + j + k$
**R** $1 + i - j - k$
**R** $1 - i - j + k$
**G** $1 - i + j - k$
$1 + i + j + k$
**B** $1 + i - j - k$

### 5.2 Quark-AntiQuark Pions.

In this $4HD$ Feynman checkerboard version of the $D_4 - D_5 - E_6$ model, pions are made up of first generation valence Quark-AntiQuark pairs.
The pion bound state of valence Quark-AntiQuark pairs has a soliton structure that would, projected onto a 2-dimensional spacetime, be a Sine-Gordon breather (see WWW URL http://www.gatech.edu/tsmith/SnGdnPion.html [13]).

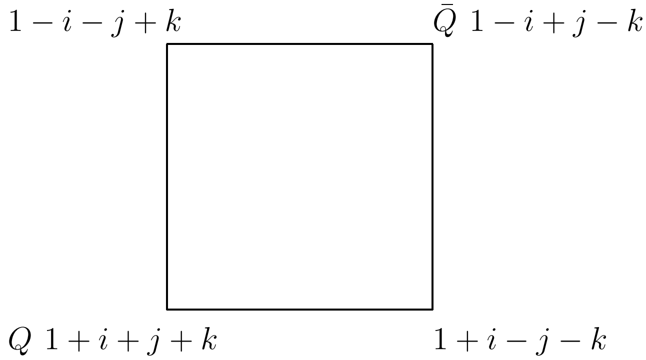
The complicated structure of the sea quarks and binding gluons can be ignored in a $4HD$ Feynman checkerboard approximation that uses the valence Quark and AntiQuark and path length $L_{313GeV}$ that is the Compton wavelength of the first generation quark constituent mass.

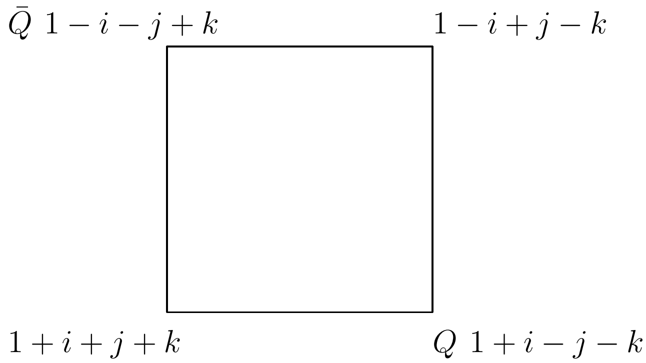
The $4HD$ structure used to approximate the pion is the 2-link Exchange Propagator.

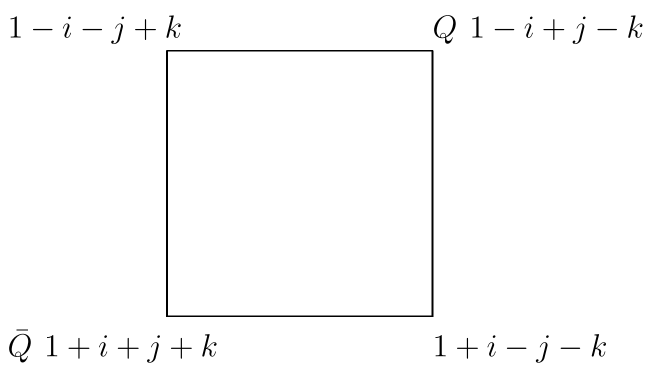
Here is a coordinate sequence representation of the approximate $4HD$ Feynman checkerboard path of a pion:

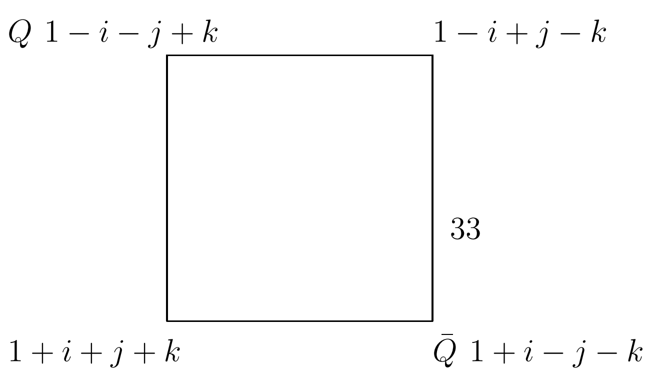
| $Quark$ | $AntiQuark$ |
|---|---|
| $0$ | $0$ |
| $1+i+j+k$ | $1-i+j-k$ |
| $2$ | $2$ |
| $3+i-j-k$ | $3-i-j+k$ |
| $4$ | $4$ |
| $5-i+j-k$ | $5+i+j+k$ |
| $6$ | $6$ |
| $7-i-j+k$ | $7+i-j-k$ |
| $8$ | $8$ |
| $9+i+j+k$ | $9-i+j-k$ |
| $10$ | $10$ |
| $11+i-j-k$ | $11-i-j+k$ |
| $12$ | $12$ |
| $13-i+j-k$ | $13+i+j+k$ |
| $14$ | $14$ |
| $15-i-j+k$ | $15+i-j-k$ |
| $16$ | $16$ |

The following page contains a Square Diagram representation of the approximate $4HD$ Feynman checkerboard path of a pion:


$1-i-j+k$
$\bar{Q}$ $1-i+j-k$
$Q$ $1+i+j+k$
$1+i-j-k$


$\bar{Q}$ $1-i-j+k$
$1-i+j-k$
$1+i+j+k$
$Q$ $1+i-j-k$


$1-i-j+k$
$Q$ $1-i+j-k$
$\bar{Q}$ $1+i+j+k$
$1+i-j-k$


$Q$ $1-i-j+k$
$1-i+j-k$
$1+i+j+k$
$\bar{Q}$ $1+i-j-k$

### 5.3 Spin-2 Physical Gravitons.

In this $4HD$ Feynman checkerboard version of the $D_4 - D_5 - E_6$ model, spin-2 physical gravitons are made up of the 4 translation spin-1 gauge bosons of the 10-dimensional $Spin(5)$ de Sitter subgroup of the 15-dimensional $Spin(6)$ Conformal group used to construct Einstein-Hilbert gravity in the $D_4 - D_5 - E_6$ model described in hep-ph/9501252 and quant-ph/9503009.

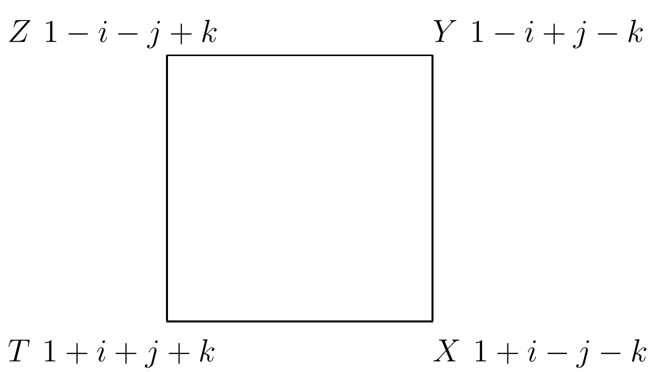
These spin-2 physical gravitons are massless, but they can have energy up to and including the Planck mass.

The Planck energy spin-2 physical gravitons are really fundamental structures with $4HD$ Feynman checkerboard path length $L_{Planck}$.

Here is a coordinate sequence representation of the $4HD$ Feynman checkerboard path of a fundamental Planck-mass spin-2 physical graviton, where $\mathbf{T}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ represent the 4 translation generator of the $Spin(5)$ de Sitter group:

| $\mathbf{T}$ | $\mathbf{X}$ | $\mathbf{Y}$ | $\mathbf{Z}$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $1+i+j+k$ | $1+i-j-k$ | $1-i+j-k$ | $1-i-j+k$ |
| $2$ | $2$ | $2$ | $2$ |
| $3+i+j+k$ | $3+i-j-k$ | $3-i+j-k$ | $3-i-j+k$ |
| $4$ | $4$ | $4$ | $4$ |
| $5+i+j+k$ | $5+i-j-k$ | $5-i+j-k$ | $5-i-j+k$ |
| $6$ | $6$ | $6$ | $6$ |
| $7+i+j+k$ | $7+i-j-k$ | $7-i+j-k$ | $7-i-j+k$ |
| $8$ | $8$ | $8$ | $8$ |

The following is a Square Diagram representation of the $4HD$ Feynman checkerboard path of a fundamental Planck-mass spin-2 physical graviton:

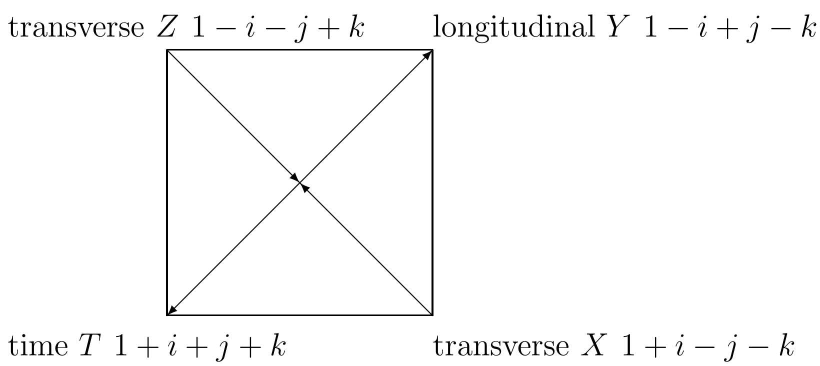



The representation above is for a timelike path at rest in space.

With respect to gravitons, we can see something new and different by letting the path move in space as well.

Let **T** and **Y** represent time and longitudinal space, and
**X** and **Z** represent transverse space.

Then, as discussed in Feynman's *Lectures on Gravitation*, pp. 41-42 [7], the Square Diagram representation shows that our spin-2 physical graviton is indeed a spin-2 particle.




Spin-2 physical gravitons of energy less than the Planck mass are more complicated composite gauge boson structures with approximate $4HD$ Feynman checkerboard path length $L_{gravitonenergy}$.

They can be deformed from a square shape, but retain their spin-2 nature as described by Feynman [7].

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